SECTION 11.11 — SAMPLE SOLUTIONS

Problem: Let $f(x) = \sin x$ and let $a = \pi/6$.

(i) Using the information given below, determine the Taylor Polynomials $T_1(x)$, $T_3(x)$ and $T_5(x)$ of $f(x) = \sin x$ centered at $a = \pi/6$.

(ii) Using an appropriate Taylor Polynomial of $f(x) = \sin x$ centered at $a = \pi/6$, find an approximation of $\sin(35^\circ)$ which is correct to 5 decimal places.

THE Given Information: $f(x) = \sin x$, $a = \pi/6$

Solution for (i).

n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$1/2$
1	$\cos x$	$\sqrt{3}/2$
2	$-\sin x$	$-1/2$
3	$-\cos x$	$-\sqrt{3}/2$
4	$\sin x$	$1/2$
5	$\cos x$	$\sqrt{3}/2$

$c_n(x-\pi/6)^n$

$$c_n(x-a)^n = \frac{f^{(n)}(a)}{n!}\left(x-\frac{\pi}{6}\right)^n$$

$$T_5(x) = \frac{(\frac{1}{2})}{0!}\left(x-\frac{\pi}{6}\right)^0 + \frac{(\frac{\sqrt{3}}{2})}{1!}\left(x-\frac{\pi}{6}\right) + \frac{(-\frac{1}{2})}{2!}\left(x-\pi/6\right)^2$$

$$+ \frac{(-\sqrt{3}/2)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{(\frac{1}{2})}{4!}\left(x-\pi/6\right)^4 + \frac{(\frac{\sqrt{3}}{2})}{5!}\left(x-\frac{\pi}{6}\right)^5$$

$$T_5(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{48}\left(x-\frac{\pi}{6}\right)^4 + \frac{\sqrt{3}}{240}\left(x-\frac{\pi}{6}\right)^5 .$$

$$T_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

$$T_1(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \pi/6\right)$$

The Solution for (2):

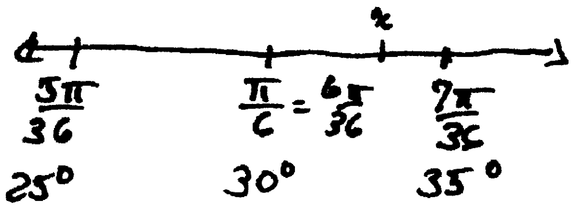
We will approximate $\sin(35^\circ)$ by approximating $f(x) = \sin(x)$ with $T_n(x)$ near $a = \pi/6$ for an appropriate degree n.

Note: The input to the $\sin x$ function actually requires that the measure x of the angle be given using RADIAN MEASURE.

Now, $\frac{\pi}{6}$ Radians $= 30^\circ = \frac{6\pi}{36}$ RADIANS.

$$35^\circ = \left(35^\circ \times \pi/180^\circ\right)\text{RADIANS} = \frac{7}{36}\pi = \frac{7\pi}{36}\text{ RADIANS}$$

ON the x-axis:



$$\left|x - \frac{\pi}{6}\right| \le d = \frac{\pi}{36}$$

$$\frac{7\pi}{36} - \frac{6\pi}{36} = \frac{\pi}{36} = d .$$

Once we discover the appropriate degree n, we will use

$$T_n(7\pi/36) \approx \sin(7\pi/36) = \sin(35^\circ)$$

We will use Taylor's Inequality (Here with $a = \pi/6$ and with $d = \pi/36$).

In this situation, Taylor's Inequality says that, for all x with $|x - \pi/6| \le \pi/36$, the error $R_n(x)$ in the approximation $T_n(x) \approx \sin x$ satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}\left|x - \frac{\pi}{6}\right|^{n+1} \le \frac{M}{(n+1)!}\left(\frac{\pi}{36}\right)^{n+1}$$

where M is a constant such that $|f^{(n+1)}(x)| \le M$ for all x with $|x - \pi/6| \le \pi/36$.

Here, because all of the higher order derivatives of $\sin x$ consist of $\pm \sin x$ or $\pm \cos x$,

$$f^{(n+1)}(x) = \pm \sin x \quad \text{or} \quad f^{(n+1)}(x) = \pm \cos x$$

for all n, and so we know that

$$|f^{(n+1)}(x)| = |\pm \sin x| \le 1 \quad \text{or} \quad |f^{(n+1)}(x)| = |\pm \cos x| \le 1$$

for all n and all x. So, we can use $M = 1$ for all degrees $(n+1)$.

Thus, in this problem,

$$|R_n(x)| \le \frac{1}{(n+1)!}\left(\frac{\pi}{36}\right)^{n+1} \quad \text{for all } n \text{ when } |x - \tfrac{\pi}{6}| \le \tfrac{\pi}{36}$$

We only need to calculate $\frac{1}{(n+1)!}\left(\frac{\pi}{36}\right)^{n+1}$

for various values of n to find a value for n sufficient to make the error be ≤ 0.000005.

We try using $n=2$ first.

Applying Taylors inequality with $n=2$, $M=1$ and $d=\frac{\pi}{36}$,

$$|R_2(x)| \leq \frac{1}{3!}\left|x-\frac{\pi}{6}\right|^3 \leq \frac{1}{3!}\left(\frac{\pi}{36}\right)^3 = 0.000110762$$

when $|x-\pi/6| \leq \pi/36$, in particular also at $x = 7\pi/36$.

So

$$|R_2(35^\circ)| = |R_2(\tfrac{7\pi}{36})| \leq 0.000110762.$$

We need $|R_n(x)| \leq 0.000005$, so $n=2$ is not appropriate.

Try $n=3$.

Applying Taylor's Inequality with $n=3$, $M=1$ and $d=\pi/36$,

$$|R_3(x)| \leq \frac{1}{4!}\left|x-\frac{\pi}{6}\right|^4 \leq \frac{1}{4!}\left(\frac{\pi}{36}\right)^4 = 0.0000002416$$

when $|x-\frac{\pi}{6}| \leq \pi/36$, in particular also at $x = 7\pi/36$.

so, $|R_3(35^\circ)| = |R_3(\frac{7\pi}{36})| \leq 0.0000002416 \leq 0.000005$

$\therefore$ Use $n=3$, and $T_3(\frac{7\pi}{36}) \approx \sin(\frac{7\pi}{36})$ correct to 5 decimal places!

$$T_3\left(\frac{7\pi}{36}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{7\pi}{36} - \frac{6\pi}{36}\right) + \frac{1}{4}\left(\frac{7\pi}{36} - \frac{6\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{7\pi}{36} - \frac{6\pi}{36}\right)^3 .$$

$$T_3\left(\frac{7\pi}{36}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3$$

$$T_3\left(\frac{7\pi}{36}\right) = 0.573575192 \approx \sin(35^\circ)$$

correct to 5 decimal places.

The calculator gives

$$\sin(35^\circ) = 0.573576436$$

Both of these numbers rounded to 5 places round to the same number:

$$\sin(35^\circ) \approx 0.57358$$ rounded to 5 places.